6.a) Find all solns of the following equations

$y'' - y' - 2y = e^{-x}$

Soln:

Given equation is $y'' - y' - 2y = e^{-x} \rightarrow (1)$

The characteristic polynomial is

$p(r) = r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r = 2, -1$

The soln are

$= c_1\varphi_1 + c_2\varphi_2$

$= c_1 e^{2x} + c_2 e^{-x} \rightarrow (2)$

i.e) $\left.\begin{aligned} \varphi_1(x) &= e^{2x} \\ \varphi_2(x) &= e^{-x} \end{aligned}\right\} \rightarrow (3)$

The particular soln ψ_p of the non-homogenous eqn is of the form

$\psi_p(x) = u_1(x)\varphi_1(x) + u_2(x)\varphi_2(x)$

$= u_1(x) e^{2x} + u_2(x) e^{-x} \rightarrow (4)$

where $u_1(x)$ and $u_2(x)$ are given by

$$u_1(x) = -\int_{x_0}^{x} \frac{\varphi_2(t)\, b(t)}{W(\varphi_1, \varphi_2)(t)}\, dt$$

$$u_2(x) = \int_{x_0}^{x} \frac{\varphi_1(t)\, b(t)}{W(\varphi_1, \varphi_2)(t)}\, dt$$

$\varphi_1(x) = e^{2x} \Rightarrow \varphi_1'(x) = 2e^{2x}$

$\varphi_2(x) = e^{-x} \Rightarrow \varphi_2'(x) = -e^{-x}$

$$W(\varphi_1, \varphi_2) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix}$$

$= -e^{2x}e^{-x} - e^{-x} \cdot 2e^{2x}$

$= -e^{x} - 2e^{2}$

$W(\varphi_1, \varphi_2) = -3e^{x}$

$$u_1(x) = -\int \frac{e^{-x}\, e^{-x}}{-3e^{x}}\, dx$$

$$= \frac{1}{3}\int e^{-3x}\, dx$$

$$= -\frac{1}{3}\left[\frac{e^{-3x}}{-3}\right]$$

$$u_1(x) = \frac{-1}{9} e^{-3x}$$

$$u_2(x) = \int \frac{e^{2x} \cdot e^{-x}}{-3e^{x}}\, dx$$

$$= -\frac{1}{3}\int \frac{e^{x}}{e^{x}}\, dx$$

$$= -\frac{1}{3}[x]$$

$$u_2(x) = -\frac{x}{3}$$

$\psi_p = u_1\varphi_1 + u_2\varphi_2$

Now, $\psi = \psi_p + c_1\varphi_1 + c_2\varphi_2 \rightarrow (*)$

$$\psi_p = u_1\varphi_1 + u_2\varphi_2$$

$$= -\frac{1}{9} e^{-3x} e^{2x} - \frac{x}{3}(e^{-x})$$

$$\psi_p = -\frac{1}{9} e^{-x} - \frac{x e^{-x}}{3}$$

$$\therefore (*) \Rightarrow \psi = -\frac{1}{9} e^{-x} - \frac{x e^{-x}}{3} + c_1 e^{2x} + c_2 e^{-x}$$

$$\psi = -\frac{e^{-x}}{3}\left[\tfrac{1}{3} + x\right] + c_1 e^{2x} + c_2 e^{-x}$$

6.b) $y'' + 4y = \cos x$

Soln: Given eqn is $y'' + 4y = \cos x$

The characteristic polynomial

$$r^2 + 4 = 0$$

$$r^2 = -4$$

$$r = \pm 2i$$

The soln is

$$= c_1 \cos 2x + c_2 \sin 2x$$

$\varphi_1(x) = \cos 2x$, $\varphi_2(x) = \sin 2x$

$\varphi_1'(x) = -2\sin 2x$, $\varphi_2'(x) = 2\cos 2x$

The particular soln

$$\psi_p(x) = u_1\varphi_1 + u_2\varphi_2$$

$$= u_1 \cos 2x + u_2 \sin 2x$$

Where u_1, u_2 given by

$$u_1 = -\int_{x_0}^{x} \frac{\varphi_2(x)\, b(x)}{W(\varphi_1,\varphi_2)(x)}\, dx$$

$$u_2 = \int_{x_0}^{x} \frac{\varphi_1(x)\cdot b(x)}{W(\varphi_1,\varphi_2)(x)}\, dx$$

$$W(\varphi_1,\varphi_2) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$W(\varphi_1,\varphi_2) = 2$$

$$u_1 = -\int \frac{\sin 2x \cos x}{2}\, dx$$

$$= -\frac{1}{4}\int [\sin 3x + \sin x]\, dx$$

$$= -\frac{1}{4}\left[-\frac{\cos 3x}{3} - \cos x\right]$$

$$u_1 = \frac{\cos 3x}{12} + \frac{\cos x}{4}$$

$$u_2 = \int \frac{\cos 2x \cos x}{2}\, dx$$

$$= \frac{1}{4}\int [\cos 3x + \cos x]\, dx$$

$$= \frac{1}{4}\left[\frac{\sin 3x}{3} + \sin x\right]$$

$$u_2 = \frac{\sin 3x}{12} + \frac{\sin x}{4}$$

$$\psi_p = u_1\varphi_1 + u_2\varphi_2$$

$$= \frac{1}{12}(3\cos x + \cos 3x)\cos 2x + \frac{1}{12}(3\sin x + \sin 3x)\sin 2x$$

$$= \frac{1}{12}[3\cos x\cos 2x + \cos 3x\cos 2x + 3\sin x\sin 2x + \sin 3x\sin 2x]$$

$$= \frac{1}{12}[3(\cos x\cos 2x + \sin x\sin 2x) + (\cos 3x\cos 2x + \sin 3x\sin 2x)]$$

$$= \frac{1}{12}[3\cos x + \cos x] = 4\frac{\cos x}{12}$$

$$\psi_p = \frac{\cos x}{3}$$

Now, $\psi = \psi_p + c_1\varphi_1 + c_2\varphi_2$

$$= \frac{\cos x}{3} + c_1\cos 2x + c_2\sin 2x.$$

2

6.c) $y'' - 7y' + 6y = \sin x$

Soln:

Given eqn is

$$y'' - 7y' + 6y = \sin x$$

The characteristic polynomial is

$$r^2 - 7r + 6 = 0$$

$$(r-1)(r-6) = 0$$

$$r = 1, 6$$

The Soln is

$$= c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

$$= c_1 e^{x} + c_2 e^{6x}$$

$$\therefore \quad \varphi_1(x) = e^{x}, \qquad \varphi_2(x) = e^{6x}$$

$$\varphi_1'(x) = e^{x}, \qquad \varphi_2'(x) = 6e^{6x}$$

The Particular soln

$$\psi_p(x) = u_1\varphi_1(x) + u_2\varphi_2(x)$$

$$= u_1 e^{x} + u_2 e^{6x}$$

where u_1, u_2 given by

$$u_1 = -\int_{x_0}^{x_1} \frac{\varphi_2(x)\, b(x)}{W(\varphi_1, \varphi_2)(x)}\, dx$$

$$u_2 = \int_{x_0}^{x_1} \frac{\varphi_1(x)\, b(x)}{W(\varphi_1, \varphi_2)(x)}\, dx$$

$$W(\varphi_1, \varphi_2)(x) = \begin{vmatrix} e^{x} & e^{6x} \\ e^{x} & 6e^{6x} \end{vmatrix}$$

$$= 6e^{x}.e^{6x} - e^{x}.e^{6x}$$

$$= 6e^{7x} - e^{7x}$$

$$W(\varphi_1, \varphi_2)(x) = 5e^{7x}$$

$$u_1 = -\int_{x_0}^{x_1} \frac{e^{6x}\sin x}{5e^{7x}}\, dx$$

$$= -\frac{1}{5}\int e^{-x}\sin x\, dx$$

$$= -\frac{1}{5}\left[\frac{e^{-x}}{1+1}(-\sin x - \cos x)\right]$$

$$u_1 = \frac{e^{-x}}{10}(\sin x + \cos x)$$

$$u_2 = \int_{x_0}^{x} \frac{e^{x}\sin x}{5e^{7x}}\,dx$$

$$= \frac{1}{5}\int e^{-6x}\sin x\,dx$$

$$= \frac{1}{5}\left[\frac{e^{-6x}}{36+1}(-6\sin x - \cos x)\right]$$

$$= \frac{1}{5}\left[\frac{e^{-6x}}{37}(-6\sin x - \cos x)\right]$$

Now,

$$\psi = \psi_p + c_1\varphi_1 + c_2\varphi_2$$

$$= \frac{e^{-x}}{10}[\sin x + \cos x]e^{x} + \frac{1}{5}\left[\frac{e^{-6x}}{37}(-6\sin x - \cos x)e^{6x}\right] + c_1e^{x} + c_2e^{6x}$$

$$= \sin x\left(\frac{1}{10} - \frac{6}{185}\right) + \cos x\left(\frac{1}{10} - \frac{1}{185}\right) + c_1e^{x} + c_2e^{6x}$$

$$= \sin x\left(\frac{5}{74}\right) + \cos x\left(\frac{7}{74}\right) + c_1e^{x} + c_2e^{6x}$$

$$= \frac{1}{74}(5\sin x + 7\cos x) + c_1e^{x} + c_2e^{6x}$$

6.d) $y'' + y = \sec x \quad (-\pi/2 < x < \pi/2)$

Soln:

Given eqn is

$$y'' + y = \sec x$$

The characteristic polynomial is

$$r^2 + 1 = 0$$

$$r^2 = -1$$

$$r = \pm i$$

The soln is

$$= e^{0x}(c_1\cos x + c_2\sin x)$$

$$= c_1\cos x + c_2\sin x$$

$\therefore \varphi_1(x) = \cos x, \quad \varphi_2(x) = \sin x$

$\varphi_1'(x) = -\sin x \quad\quad \varphi_2'(x) = \cos x$

The particular soln is

$$y_p = u_1\varphi_1 + u_2\varphi_2$$

$$u_1 = -\int \frac{\varphi_2(x)\, b(x)}{W(\varphi_1,\varphi_2)(x)}\, dx$$

$$u_2 = \int \frac{\varphi_1(x)\, b(x)}{W(\varphi_1,\varphi_2)(x)}\, dx$$

$$W(\varphi_1,\varphi_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$= 1$$

$$u_1 = -\int \sin x \sec x \, dx$$

$$= -\int \sin x \cdot \frac{1}{\cos x}\, dx = -\int \tan x \, dx$$

$$= -\log \sec x$$

$$= \log (\sec x)^{-1} = \log\left(\frac{1}{\sec x}\right)$$

$$u_1 = \log \cos x$$

$$u_2 = \int \cos x \cdot \sec x \, dx$$

$$= \int \cos x \cdot \frac{1}{\cos x}\, dx$$

$$= \int dx$$

$$u_2 = x$$

Now, $y = y_p + c_1 \cos x + c_2 \sin x$

$$y = (\log \cos x)\cos x + x \sin x + c_1 \cos x + c_2 \sin x$$

6.e) $y'' + 9y = \sin 3x$

Soln:

Given eqn is $y'' + 9y = \sin 3x$

The characteristic polynomial is

$$r^2 + 9 = 0$$

$$r^2 = -9$$

$$r = \pm 3i$$

The soln is

$$= e^{0x}(c_1 \cos 3x + c_2 \sin 3x)$$

$$= c_1 \cos 3x + c_2 \sin 3x$$

$\therefore$ $\varphi_1(x) = \cos 3x$, $\varphi_2(x) = \sin 3x$

$\varphi_1'(x) = -3\sin 3x$, $\varphi_2'(x) = 3\cos 3x$

The particular soln is

$$\psi_p = u_1\varphi_1 + u_2\varphi_2$$

$$u_1 = -\int_{x_0}^{x_1} \frac{\varphi_2(x)\, b(x)}{W(\varphi_1,\varphi_2)(x)} dx$$

$$u_2 = \int \frac{\varphi_1(x)\, b(x)}{W(\varphi_1,\varphi_2)(x)} dx$$

$$W(\varphi_1,\varphi_2) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$= 3\cos^2 3x + 3\sin^2 3x$$

$$= 3(\cos^2 3x + \sin^2 3x)$$

$$W(\varphi_1,\varphi_2) = 3$$

$$u_1 = -\int \frac{\sin 3x \sin 3x}{3} dx$$

$$= -\frac{1}{3\times 2}\int(\cos(0)x - \cos 6x)dx$$

$$= -\frac{1}{6}\int(1-\cos 6x)dx$$

$$u_1 = -\frac{1}{6}\left[x - \frac{\sin 6x}{6}\right]$$

$$u_2 = \int \frac{\cos 3x \sin 3x}{3} dx$$

$$= \frac{1}{6}\int(\sin 6x - \sin(0)x)\, dx$$

$$= \frac{1}{6}\int \sin 6x\, dx$$

$$= \frac{1}{6}\left(-\frac{\cos 6x}{6}\right)$$

$$u_2 = -\frac{1}{36}\cos 6x$$

Now,

$$\psi = \psi_p + c_1 \cos 3x + c_2 \sin 3x$$

$$= -\frac{1}{6}\left[x - \frac{\sin 6x}{6}\right]\cos 3x - \frac{1}{36}\cos 6x \sin 3x + c_1 \cos 3x + c_2 \sin 3x$$

$$= -\frac{x\cos 3x}{6} + \frac{\sin 6x \cos 3x}{36} - \frac{\cos 6x \sin 3x}{36} + c_1 \cos 3x + c_2 \sin 3x$$

$$= -\frac{x}{6}\cos 3x + \frac{1}{36}\left[\frac{\sin 9x + \sin 3x}{2} - \left(\frac{\sin 9x - \sin 3x}{2}\right)\right] + c_1 \cos 3x + c_2 \sin 3x$$

$$= -\frac{x}{6}\cos 3x + \frac{1}{36}\left[2\frac{\sin 3x}{2}\right] + c_1 \cos 3x + c_2 \sin 3x$$

$$= -\frac{x}{6}\cos 3x + \frac{\sin 3x}{36} + c_1 \cos 3x + c_2 \sin 3x.$$

6.f) $y'' + y = \tan x \quad (-\pi/2 < x < \pi/2)$

Soln:

Given eqn is $y'' + y = \tan x$

The characteristic polynomial

$$r^2 + 1 = 0$$
$$r^2 = -1$$
$$r = \pm i$$

The soln is $= c_1 \cos x + c_2 \sin x$

$\varphi_1(x) = \cos x$, $\varphi_2(x) = \sin x$

$\varphi_1'(x) = -\sin x$, $\varphi_2'(x) = \cos x$

The Particular soln is

$$\psi_p = u_1\varphi_1 + u_2\varphi_2$$
$$= u_1 \cos x + u_2 \sin x$$

$$u_1 = -\int_{x_0}^{x_1} \frac{\varphi_2(x)\, b(x)}{W(\varphi_1, \varphi_2)(x)}\, dx$$

$$u_2 = \int_{x_0}^{x_1} \frac{\varphi_1(x)\, b(x)}{W(\varphi_1, \varphi_2)(x)}\, dx$$

$$W(\varphi_1, \varphi_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$= \cos^2 x + \sin^2 x$

$W(\varphi_1, \varphi_2) = 1$

$u = \tan x \quad \int dv = \int \sin x$

$du = \sec^2 x \, dx \quad v = -\cos x$

$$u_1 = -\int \frac{\sin x \tan x}{1} dx$$

$$= -\left[\tan x(-\cos x) + \int \cos x \sec^2 x \, dx\right]$$

$$= -\left[-\frac{\sin x}{\cos x} \cdot \cos x + \int \cos x \cdot \frac{1}{\cos^2 x} dx\right]$$

$$= -\left[-\sin x + \int \frac{1}{\cos x} dx\right]$$

$$= -\left[-\sin x + \int \sec x \, dx\right]$$

$$= -\left[-\sin x + \log(\sec x + \tan x)\right]$$

$$u_1 = \sin x - \log(\sec x + \tan x)$$

$$u_2 = \int \cos x \tan x \, dx$$

$$= \int \cos x \cdot \frac{\sin x}{\cos x} dx$$

$$= \int \sin x \, dx$$

$$u_2 = -\cos x$$

Now,

$$\varphi = [\sin x - \log(\sec x + \tan x)]\cos x - \cos x \sin x + c_1 \cos x + c_2 \sin x$$

$$= \sin x \cos x - \cos x \log(\sec x + \tan x) - \cos x \sin x + c_1 \cos x + c_2 \sin x$$

$$= -\cos x \log(\sec x + \tan x) + c_1 \cos x + c_2 \sin x.$$

6.g) $y'' + y = x$

Soln:

Given eqn is $y'' + y = x$

The characteristic polynomial is

$$r^2 + 1 = 0$$

$$r^2 = -1$$

$$r = \pm i$$

Soln is

$$= c_1 \cos x + c_2 \sin x$$

$\varphi_1(x) = \cos x$, $\varphi_2(x) = \sin x$

$\varphi_1'(x) = -\sin x$, $\varphi_2'(x) = \cos x$

49 5

The particular soln is

$$\psi_p = u_1\varphi_1 + u_2\varphi_2$$

$$= u_1\cos x + u_2\sin x$$

$$u_1 = -\int \frac{\varphi_2(x)\, b(x)}{W(\varphi_1,\varphi_2)(x)}\, dx$$

$$u_2 = \int \frac{\varphi_1(x)\, b(x)}{W(\varphi_1,\varphi_2)(x)}\, dx$$

$$W(\varphi_1,\varphi_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$W(\varphi_1,\varphi_2) = 1$$

$$u_1 = -\int \frac{x\sin x}{1}\, dx$$

$$= -\int x\sin x\, dx$$

$$= -[x(-\cos x) - (-\sin x)]$$

$$= -[-x\cos x + \sin x]$$

$$u_1 = x\cos x - \sin x$$

$$u_2 = \int \cos x \cdot x\, dx$$

$$= \int x\cos x\, dx$$

$$= [x\sin x - (-\cos x)]$$

$$u_2 = x\sin x + \cos x$$

Now, $\psi = \psi_p + c_1\cos x + c_2\sin x$

$$= (x\cos x - \sin x)\cos x + (x\sin x + \cos x)\sin x + c_1\cos x + c_2\sin x$$

$$= x\cos^2 x - \sin x\cos x + x\sin^2 x + \cos x\sin x + c_1\cos x + c_2\sin x$$

$$= x(\cos^2 x + \sin^2 x) + c_1\cos x + c_2\sin x$$

$$\psi = x + c_1\cos x + c_2\sin x$$

6.h) $y'' - 2y' = e^{x}\sin x$

Soln:

Given eqn is $y''-2y' = e^{x}\sin x$

The characteristic polynomial is

$$r^2 - 2r = 0$$

$$r(r-2) = 0$$

$$r = 0, 2$$

Soln

$$= c_1 e^{0x} + c_2 e^{2x}$$

$$= c_1 + c_2 e^{2x}$$

$\varphi_1 = 1$, $\varphi_2 = e^{2x}$

$\varphi_1' = 0$, $\varphi_2' = 2e^{2x}$

The Particular soln is

$$\psi_p = u_1\varphi_1 + u_2\varphi_2$$

$$= u_1 + u_2 e^{2x}$$

$$u_1 = -\int \frac{\varphi_2(x)\, b(x)}{W(\varphi_1, \varphi_2)(x)}\, dx$$

$$u_2 = \int \frac{\varphi_1(x)\, b(x)}{W(\varphi_1, \varphi_2)(x)}\, dx$$

$$W = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix}$$

$$W = 2e^{2x}$$

$$u_1 = -\int \frac{e^{2x} e^{x}\sin x}{2e^{2x}}\, dx$$

$$= -\frac{1}{2}\int e^{x}\sin x\, dx$$

$$= -\frac{1}{2}\left[\frac{e^{x}}{2}(\sin x - \cos x)\right]$$

$$u_1 = -\frac{e^{x}}{4}(\sin x - \cos x)$$

$$u_2 = \int \frac{e^{x}\sin x}{2e^{2x}}\, dx$$

$$= \frac{1}{2}\int e^{-x}\sin x\, dx$$

$$= \frac{1}{2}\left[\frac{e^{-x}}{2}(-\sin x - \cos x)\right]$$

(51) (6)

$$u_2 = \frac{e^{-x}}{4}(-\sin x - \cos x)$$

Now,

$$y = y_p + c_1 + c_2 e^{2x}$$

$$= -\frac{e^{x}}{4}(\sin x - \cos x) + \frac{e^{-x}}{4}(-\sin x - \cos x)e^{2x} + c_1 + c_2 e^{2x}$$

$$= -\frac{e^{x}}{4}\sin x + \frac{e^{x}}{4}\cos x - \frac{e^{x}}{4}\sin x - \frac{e^{x}}{4}\cos x + c_1 + c_2 e^{2x}$$

$$= -\frac{2e^{x}}{4}\sin x + c_1 + c_2 e^{2x}$$

$$y = -\frac{e^{x}}{2}\sin x + c_1 + c_2 e^{2x}$$

6. i) $y'' + 2iy' + y = x$

j) $y'' - 4y' + 5y = 3e^{-x} + 2x^2$

k) $y'' + y = 2\sin x \sin 2x$

l) $y'' + y = \sec x \; (-\pi/2 < x < \pi/2)$

m) $4y'' - y = e^{x}$.

n) $6y'' + 5y' - 6y = x$.